Lesson 3 :

Exercises with sequences and series

We saw in the previous lessons that sequences are important in two respects

1. They enable us to understand notions of elementary topology on rationals and reals, which shed light on these two sets.
2. They are essential to introduce the fundamental notions of continuity of certain real functions, and also to calculate, when they have one, their derivative function.

Regarding point 1 : in a word, rationals, like reals, are dense sets. However, rationals have “holes”, while reals do not. The set $\mathbb{R}$ of reals is complete. The set $\mathbb{Q}$ of rationals is not. This is the main reason why the set of choice for working with numbers is the set of reals. And to support the intuition – for those of us who have a geometric mental view of things – we have the line of numbers that represents all numbers, from natural numbers to real numbers.

Regarding point 2 : continuous and differentiable functions are a fundamental tool in mathematics and physics. To understand them well, we must first understand sequences of numbers.

We saw that there is no essential difference between sequences and series. They are two different ways of looking at the same question : the convergence or not of a collection of numbers towards a limit.

For example, the sequence

$$1 \quad 1.5 \quad 1.75 \quad 1.875 \quad 1.9375 \quad 1.96875 \quad \text{etc.} \tag{3.1}$$

and the series

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + ... \tag{3.2}$$

are two different ways of looking at the same thing.

In our opinion, series are more convenient for studying convergence problems, because their construction, which remains explicit, better reveals how they are likely to behave.

This is particularly true of power series, which are a great tool for studying certain functions. The general form of a power series is

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ... + a_n x^n + ... \tag{3.3}$$

where the independent variable is x, and the a_i's are a countably infinite collection of coefficients.

To use them, we must first understand the conditions on the coefficients and the values of the independent variable x under which they are meaningful, i.e., when they converge.

Exercise 3.1 : Show, without laborious calculations, that

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} = 1.96875 \tag{3.4}$$

Hint : Use the formula learned in middle school [1] expressing more simply the sum

$$1 + a + a^2 + a^3 + a^4 + a^5 \tag{3.5}$$

We will no longer deal with topology in this 11th grade math course. We will talk about it a little more in the final year of high school. But it will mainly be a subject for readers who will continue studying mathematics in college. [2]

Lesson 3, which is easier than the previous two but equally fundamental, is devoted to exercises with sequences and series. Some exercises are solved within the lesson text, while others are provided for readers to deepen and strengthen their understanding. We don't give the solution of the latter, but we give hints, that often amount to the solution :-) A note upfront to avoid having to repeat it later : serious reading of this book requires completing *all the exercises* and even creating some of your own.

We begin with two classic exercises from middle school level : the first suitable for early middle school, and the second for the later years.

1. We saw it for instance in *Middle school mathematics : 8th and 9th grades*, Eagle's Beak Press, fall 2024.
2. A good introduction to topology is chapter 18 by Pavel Aleksandrov, in the book by Aleksandrov, Kolmogorov, and Lavrent'ev, mentioned in the suggested reading at the end of lesson 1.

Exercise 3.2 : Show that the sum of the first n positive natural numbers is

$$1 + 2 + 3 + ... + n = \frac{n(n+1)}{2} \tag{3.6}$$

Hint : See the demonstration by Carl Friedrich Gauss (1777, 1855) in exercise I.8.10, of the book *Middle school mathematics : 6th and 7th grades*, Eagle's Beak Press, fall 2024.

Alternatively, prove formula (3.6) using mathematical induction [3] (see next exercise to review what mathematical induction consists of).

Exercise 3.3 : Show that the sum of the first n squares of positive natural numbers is

$$1 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{3.7}$$

Hint : Construct a mathematical induction argument. Here is how to do it :

1. Verify that the formula is true for $n = 1$.
2. Assume that the formula is true up to n and then show that it is still true for $(n + 1)$.

   In other words, show that

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)(n+2)[2(n+1)+1]}{6} \tag{3.8}$$

   To do this, start by eliminating $(n+1)$ from both sides. Then multiply both sides by 6. Finally show that, after these simplification and multiplication by 6, the two sides are each equal to $2n^2 + 7n + 6$.

3. Also called recurrence argument, or reasoning by recurrence.

Here is a third classic exercise from middle school :

Exercise 3.4 : Show that the sum of the inverses of the natural numbers diverges to plus infinity. We note it

$$\lim_{n \to +\infty} \left\{ 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} \right\} = +\infty \tag{3.9}$$

The following results are beyond the 11th grade high school level and are included for mathematical culture :

1. The sum of the inverses of prime numbers diverges towards plus infinity.
2. The sum of the inverses of the squares of positive integers converges towards $\frac{\pi^2}{6}$. That is,

$$\lim_{n \to +\infty} \left\{ 1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2} \right\} = \frac{\pi^2}{6} \tag{3.10}$$

This second result is called the "Basel problem". It was solved in the 18th century by Leonhard Euler (1707, 1783) who, like other people who looked at the problem, was from Basel. But Euler's proof was not completely rigorous. We can say that it was heuristic in a way. The impeccable proof of the Basel problem calls upon advanced results of the theory of series that were not established until the 19th century.

Fibonacci sequence

It is a famous sequence defined by a recurrence formula after initializing it with the values of its first two terms.

$$u_0 = 1$$

$$u_1 = 1$$

then

$$u_{n+1} = u_n + u_{n-1} \tag{3.11}$$

The first fifteen terms of the Fibonacci sequence are given in table 3.1.

Table 3.1 : First fifteen terms of the Fibonacci sequence.

n	u_n
0	1
1	1
2	2
3	3
4	5
5	8
6	13
7	21
8	34
9	55
10	89
11	144
12	233
13	377
14	610
15	987

Historical note : Fibonacci (c. 1170, c. 1250), also known as Leonardo of Pisa, was the son of a merchant from Pisa. He spent several years in Bougie on the Algerian coast, where his father traded with Arab merchants of the Maghreb. During his time there, Fibonacci learned Arabic mathematics, including Al-Khwarizmi's algebra, the use of positional notation in base 10, and elements of double-entry bookkeeping.

Upon returning to Italy, he published a book in 1202 titled *Liber Abaci*. This work holds a prominent place in the history of mathematics in Europe as it was the first to provide a detailed explanation of Arabic numerals and the calculation techniques associated with them. These numerals were far more convenient for performing arithmetic operations compared to the Roman numerals that had been used so far.

Given that accounting makes extensive use of arithmetic operations (especially addition and subtraction), it is not surprising that Fibonacci's book also introduced early accounting techniques – in particular double-entry bookkeeping – learned from the Arabs, who had in turn learned them from the Indians. [4]

4. The author's website offers a free course in double-entry bookkeeping at `https://www.lapasserelle.com/online_courses/accounting`

The Fibonacci sequence obviously diverges, but it has many interesting properties. Let's study the main one. Let the sequence v_n be constructed from the Fibonacci sequence u_n in the following manner :

$$v_n = \frac{u_{n+1}}{u_n} \tag{3.12}$$

Then the following result holds :

$$\lim_{n \to +\infty} v_n = \frac{1+\sqrt{5}}{2} \tag{3.12}$$

In other words, the sequence of ratios u_{n+1}/u_n converges to the *golden ratio*.

Proof :

Before diving into the proof, let's start by examining the first terms of the sequence v_n, as recommended, to guide our approach.

Table 3.2 : First twenty terms of the sequence v_n.

n	u_n	v_n
0	1	1
1	1	2
2	2	1,5
3	3	1,6666666667
4	5	1,6
5	8	1,625
6	13	1,6153846154
7	21	1,619047619
8	34	1,6176470588
9	55	1,6181818182
10	89	1,6179775281
11	144	1,6180555556
12	233	1,6180257511
13	377	1,6180371353
14	610	1,6180327869
15	987	1,6180344478
16	1597	1,6180338134
17	2584	1,6180340557
18	4181	1,6180339632
19	6765	1,6180339985
20	10946	

How does this sequence behave ? It is neither strictly increasing nor strictly decreasing. Let's draw it :

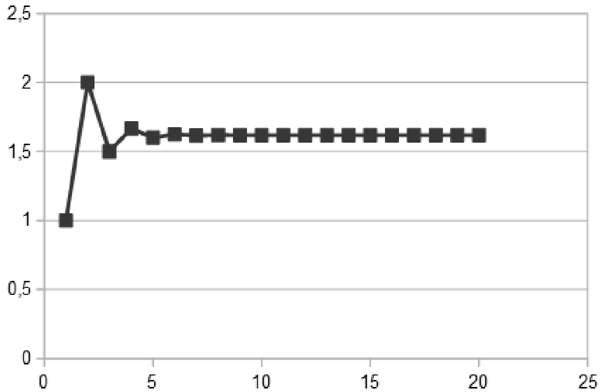

Figure 3.1 : Graph of the sequence of ratios $v_n = \frac{u_{n+1}}{u_n}$.

We see in figure 3.1 that the sequence v_n alternates by taking values above and below a limit around 1.6. This is naturally in no way a demonstration. On the other hand, it guides us to prove the convergence of v_n.

There are several ways to do this. One idea that comes to mind is to show that the terms with *even index* of the sequence v_n increase while remaining bounded. This will allow us to assert, by virtue of the completeness of the set $\mathbb{R}$, that this sequence of even terms converges to a limit. Then we will show that the limit of the even terms is $(1+\sqrt{5})/2$. Finally we will do the analog with odd terms.

Let us start by observing that $v_0 = 1$. Then we have

$$\begin{aligned} v_n &= \frac{u_{n+1}}{u_n} \\ v_{n+1} &= \frac{u_{n+2}}{u_{n+1}} \\ v_{n+2} &= \frac{u_{n+3}}{u_{n+2}} \end{aligned}$$

We can rewrite v_{n+2} as follows :

$$\begin{aligned} v_{n+2} &= \frac{u_{n+2}+u_{n+1}}{u_{n+2}} \\ &= 1+\frac{u_{n+1}}{u_{n+2}} \\ &= 1+\frac{u_{n+1}}{u_{n+1}+u_n} \\ &= 1+\frac{1}{\frac{u_{n+1}+u_n}{u_{n+1}}} \\ &= 1+\frac{1}{1+\frac{u_n}{u_{n+1}}} \\ &= 1+\frac{1}{1+\frac{1}{v_n}} \end{aligned}$$

Or equivalently

$$v_{n+2} = \frac{2v_n+1}{v_n+1} \tag{3.13}$$

Let's check our calculations. Given that $v_0 = 1$, it yields

$$v_2 = 1 + \frac{1}{2} = 1,5$$

$$v_4 = 1 + \frac{1}{1+\frac{1}{1,5}} = 1 + \frac{1,5}{2,5} = \frac{4}{2,5} = 1,6$$

All is fine.

Let us move on to the demonstration that v_n increases for even indices and decreases for odd indices. As indicated by formula (3.13), the term v_{n+2} is obtained from v_n by applying to v_n the function

$$y = \frac{2x+1}{x+1} \tag{3.14}$$

Let us try to understand why when we start with $x = 1$, we get an increasing sequence. But when we start with $x = 2$ we get a decreasing sequence.

The joint graph of the two curves $y = \frac{2x+1}{x+1}$ and $y = x$ reveals what's going on.

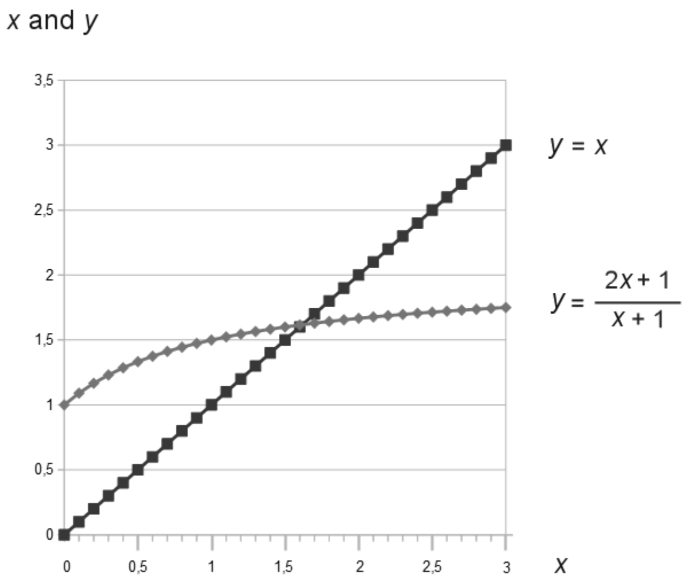

Figure 3.2 : Graphs of $y = \frac{2x+1}{x+1}$ and $y = x$.

As long as x is below the intersection point of the two curves, y is larger than x, and y remains below the intersection point. When x is above the intersection point, y is smaller than x, but remains above the intersection point.

What is the abscissa of the intersection point (which is also its ordinate) ?

We must solve the equation

$$x = \frac{2x+1}{x+1} \tag{3.15}$$

We leave it to the reader to show that the crossing is at the point with coordinates $x = y = \dfrac{1+\sqrt{5}}{2}$. This number, which is called the golden ratio, is generally denoted with the symbol ϕ.

For our demonstration to be mathematically complete, we must demonstrate logically, and not just show in figure 3.2, that when $x < \phi$, we have $x < y < \phi$. And when $\phi < x$, we have $\phi < y < x$. We also leave it as an exercise.

In summary, the sequence of v_n's with even index increases and is bounded. (It is bounded in particular by the abscissa of the crossing point.) So it converges to a limit l. A bit of epsilonite, as we did on page 27, shows that this limit necessarily satisfies the equation giving v_{n+2} in terms of v_n. This is precisely equation (3.15). Therefore the limit of the sequence of even terms of v_n is the golden ratio. Its value, with a few decimals, is

$$\phi = 1.61803... \tag{3.16}$$

Similarly, the odd-index terms of v_n decrease and are bounded from below. Therefore they have a limit, and this limit is also ϕ.

Q.E.D.

The demonstration we have just presented does not claim to be the only one, nor the most elegant. Rather, it aims to illustrate how to tackle a problem : think it through, choose an approach, take the right tools out of your toolbox, and valiantly attack the problem. Sometimes the approach works ; sometimes it doesn't, and we have to try a different path. Here, it worked.

Doing mathematics is not about memorizing a large set of tricks to solve various problems, as is still too often emphasized in some schools. Our goal is not to produce "Putnam competition champions"[5] who secure spots at top national universities, but to train future mathematicians, physicists, and engineers who will have to deal with mathematical problems and will need to apply sound methodologies.

It is also worth noting that some 11th grade math cramming manuals include exercises that seem more appropriate for 6th grade level. Students who rely solely on these books and related courses might get good grades throughout their secondary schooling. However, they will struggle with college admissions and, if they get into college, with college studies, because, despite criticisms of an overemphasis on memorizing tricks rather than developing methodology, earning a college degree in mathematics requires more effort and a deeper understanding than what these 11th grade textbooks, often filled with middle school-level exercises, can provide.[6]

5. Although the author did earn a third prize at the International Mathematical Olympiad in Hungary many years ago.

6. For example : consider an arithmetic sequence u_n whose step is equal to 7. Starting from u_0, a positive number less than 10, we reach $u_n = 61$. What is u_0 ? And for which index n do we reach $u_n = 61$?

To take a break from studying the Fibonacci sequence, let's revisit the Basel problem, see formula (3.10).

This time, we'll let a computer do the work for us, and reveal some insights. In other words, we will engage in what is called "experimental mathematics".

Exercise 3.5 : The sequence of partial sums of the inverses of the squares of positive integers is clearly increasing. We are told that it has a limit, which is $\pi^2/6$. We are not going to prove this. [7]

Our aim is to see if it converges reasonably quickly.

Up to what index n must we go so that the square root of six times the partial sum

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2} \tag{3.17}$$

exceeds the number 3.14 ?

Hint : You should find that we must go up to nearly $n = 600$.

Thales' theorem revisited and extended to irrational numbers

To conclude this lesson devoted to a first visit of sequences, series, and convergence, at work in simple situations, let us prove Thales' theorem for an irrational ratio.

Consider a triangle ABC. We recall that we proved Thales' theorem when we place a point E on AB at a distance p/q times AB from A, and a point F on AC at a distance p/q times AC from A, see figure 3.3.

Then the segment EF is parallel to BC and of length equal to p/q times BC.

But we have not demonstrated Thales' theorem when instead of a rational number p/q, we make the same construction with a positive irrational number λ.

7. To show that the limit is $\pi^2/6$ is not easy – particularly if we want to be rigorous. However to show that the sequence has a limit *is* easy.

With a little elementary calculus, which we will learn next year, it is easy to show that the sequence (3.17) is bounded. Therefore it has a limit. On the other hand, as said, showing that this limit is $\pi^2/6$ is more difficult.

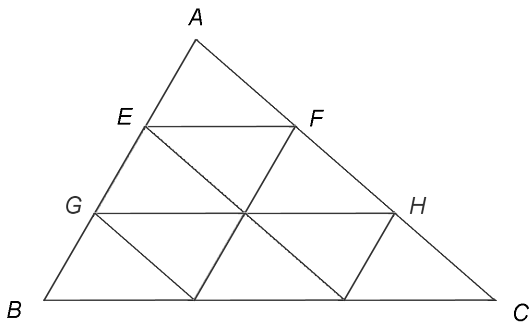

Figure 3.3 : Thales' theorem when using a rational number to split AB and AC. (In the figure we used 1/3.)

What happens if we use a split with an irrational number λ?

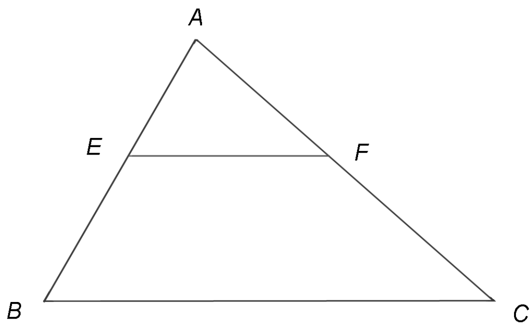

Figure 3.4 : Study of the case where $AE/AB = AF/AC = \lambda$, and lambda is irrational.

We can frame λ as precisely as we want between p/q and $(p+1)/q$. This allows us to position two points E_1 and E_2 framing E on the segment AB. We do the same with F_1 and F_2 on the segment AC.

The points E_1, E_2, F_1 and F_2 define a horizontal band. The segment EF (which a priori could be slightly inclined) is contained in this strip. Given that we can take a strip as thin as we want, this is only possible if EF is horizontal.

Analogously, EF has a length intermediate between that of E_1F_1 and that of E_2F_2. We can make these two lengths as close as we want to λ times BC. So the only possible length for EF is λ times BC.

Q.E.D.

Thales' theorem was obvious for a rational partition of AB and AC. Now we have extended it to irrational partitions. This shows the power of reasoning using the concept of convergence.

Reading suggestions

Flegg Graham, *Numbers : Their History and Meaning*, Dover, 2013.

Lesson 4 :
Continuous functions

To understand the importance of continuous functions in mathematics and physics, it is useful to briefly review the history of the development of mathematics. This history can be divided into four periods :

1. From the Egyptians and Babylonians to Euclid (-300) *Ancient mathematics* : accumulation of facts and rules in elementary geometry and arithmetic, lacking organization or a broader perspective. However, by the end of this period, Euclid wrote his influential work, *Elements*, which presented geometry in an axiomatic form. The mathematics taught in primary school and used in everyday life largely originates from this era.
2. From Euclid to Descartes (1596, 1650) : *mathematics of constant quantities*. In this long period, we can distinguish several parts :
   - First, at the end of Greek Antiquity, the marvelous school of Alexandria (from -300 to +400) which included, in addition to Euclid, Eratosthenes (c. -276, c. -194), Hero (first century AD), Diophantus, Pappus, Hypatia, etc.
   - Eastern mathematics (of Indians, Persians, Arabs) : numeration, trigonometry, and algebra. This period spans from 500 to 1500 CE. Mathematics from this era, along with much knowledge of mathematics from Antiquity, was transmitted to Europe by the Arabs. This transfer played a significant role in Europe's intellectual renaissance, which had waned in scientific and most cultural pursuits after the fall of the Western Roman Empire. The Catholic Church had taken over from the Greco-Roman world in preserving spiritual activities, but it was not oriented towards the sciences. This is a vast and fascinating story that cannot be described in a few lines

- Beginning of European mathematics : significant progress in algebra by the Italians in the 16th century, the discovery of complex numbers, and the mathematics of cosmography – which originally began at the start of the Christian era with Claudius Ptolemy (c. +100, c. +168) but underwent a revolution with Nicolas Copernicus (1473, 1543) and then Johannes Kepler (1571, 1630) – as well as advances in cartography, Descartes' analytic geometry, and more.

3. From Descartes to the beginning of the 19th century : *mathematics of variable quantities.* This period marks the triumph of European mathematics, characterized by the introduction of functions, differential and integral calculus, and the application of mathematics to physics with remarkable success, including fields like fluid mechanics and electromagnetism. Once again, it would require volumes to be described fully.
4. From the beginning of the 19th century to the present day : *contemporary mathematics.* They begin with a deepening and consolidation of the basic concepts in differential and integral calculus. It is the work of our friend Cauchy who defined clearly, as we have seen, the notions of convergence and limit. [8] Then came non-Euclidean geometry, set theory, the theory of algebraic structures, and other areas, which the reader will discover if they continue to study math in college. [9]

Functions are the fundamental concept of the period of mathematics of variable quantity. Among functions, continuous functions – and in particular differentiable ones, which we will see in the next lesson – are the most important.

It is important to understand that the idea of doing mathematics with functions represents a considerable conceptual leap compared to previous mathematics. From the Greeks until the end of the 16th century in Europe, mathematics dealt with "concrete" things. These were sometimes explicitly stated in the works of mathematicians, but they were always present implicitly : they were num-

8. Cauchy would already deserve his place in the pantheon of mathematicians for this work, but his most important works concern functions and complex analysis that he initially developed essentially alone.

9. The three-volume work, by Aleksandrov, Kolmogorov and Lavrent'ev, *Mathematics*, Dover, 1999, offers a clear, pedagogical, effective introduction, free from any pedantry (and avoiding "modern math" like the plague), to the mathematics of periods 3 and 4 above.

bers, constant quantities (possibly unknown that had to be found), points and geometric figures.

The idea that a *function*, that is to say a relation between two variables, could be a mathematical entity in itself, which could be called f, and on which one could work, was foreign to the Greek mind. The idea that an entire function could be the unknown in a riddle was inconceivable before the 18th century.

The history of the emergence of the concept of function in the 17th century is very interesting. Let us mention some important milestones.

We have seen that mathematicians were already interested in equations in early Antiquity. In the second millennium BC, the Babylonians were posing problems like this one (put in modern form) [10] : find the number x such that $x^2 + x = 3/4$. And they gave the answer – with its formula! They considered that there was only one solution, because they eliminated the negative one.

Exercise 4.1 : Solve algebraically the equation

$$x^2 + x = \frac{3}{4} \tag{4.1}$$

Draw the curve $y = x^2 + x - 3/4$.

We have seen that Al-Khwarizmi, in the 9th century of our era, invented algebra to solve all kinds of equations.

The Italians of the 16th century (Del Ferro, Tartaglia, Ferrari) discovered general solutions for polynomial equations of the third and fourth degrees in one variable.

Exercise 4.2 : Solve geometrically the equation

$$x^3 + x^2 = \frac{3}{4} \tag{4.2}$$

How many real solutions are there ?

Explain why a third-degree polynomial equation with one unknown always has at least one real solution.

10. See lesson 4 of our book, *High school mathematics : 10th grade.*

At the end of the 16th century, mathematicians were also interested in equations involving *two unknowns*, x and y. They knew that in general to determine the value of each unknown, we needed to have two equations, that is, two constraints.

For example, if I tell you that Judy is twice as old as Emily, and in three years she will be only one and a half times as old as Emily, you can determine Judy's age and Emily's age today.

Exercise 4.3 : What are Judy and Emily's ages today ?

But if I simply tell you that in one year Robert will be the same age as the square of Cecilia's age then, and that in this problem we are not considering whole numbers of years, but that the unknowns are real numbers measuring durations, you cannot determine Robert's age and Cecilia's age. If we call y Robert's age today, and x Cecilia's age, we can transform our problem into the equation

$$(x+1)^2 = y+1 \tag{4.3}$$

Since we cannot solve this equation, until around 1620 mathematicians simply considered that it was uninteresting.

René Descartes took a different view of the question. He said

– No, no, equation (4.3) is very interesting ! Of course, we cannot "solve" it, that is, we cannot find x and y. But equation (4.3) establishes a *relation* between x and y.[11]

Descartes, as is known, created Cartesian coordinate systems and analytic geometry. For Cartesian coordinate systems, he was inspired by cartographers who had already been locating points on the spherical globe using latitude and longitude for a century.

When it comes to "drawing a curve", predecessors to Descartes can also be found. Oresme (c. 1320, 1382) plotted the speed of a horse rider against time on a two-dimensional graph, with speed as the ordinate and time as the abscissa. Oresme even pointed out that the area under the curve between two time points represented the distance covered by the rider. More on this in lesson 5.

Other mathematicians, such as Pierre de Fermat (c. 1605, 1665), contributed to the birth of analytic geometry. But it was Descartes who really showed and began to exploit its full power.

11. The term *function* was introduced a little later by Gottfried Leibniz. And it has a slightly less general meaning than "relation".

Descartes thus "represented" the relation (4.3) as this :

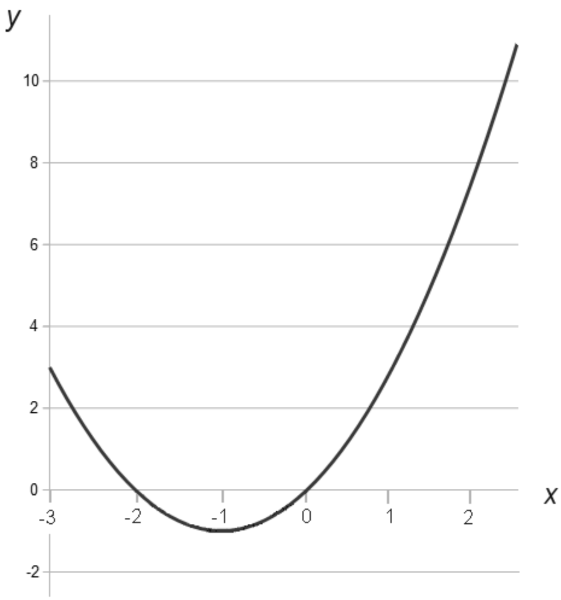

Figure 4.1 : Graph of the relation $y = (x+1)^2 - 1$.

Furthermore Descartes explained that algebraic problems could be transformed into geometric problems, and vice versa.

Exercise 4.4 : Solve the following problem geometrically. Find the values of x and y satisfying the two equations :

$$\begin{aligned} (x+1)^2 &= y + 1 \\ x + 4 &= \frac{2}{3}y \end{aligned} \tag{4.4}$$

Relations where each x corresponds to a single y are called functions. But the concept of a relation as introduced by Descartes is broader.

Exercise 4.5 : Plot the locus of all the points in the plane whose coordinates $(x,\ y)$ satisfy the constraint :

$$2x^2 + y^2 - xy + 3x + y - 7 = 0 \tag{4.5}$$

To which family does this curve belong ?

Explain why it is not the graph of a function.

Hint : For a relation to qualify as a function, to each value of the independent variable x there must be associated only one value of the dependent variable y.

Why the concept of function became fundamental in the 17th century

The history of the world explains it. After the disastrous scholasticism of the 13th century, Europe began to think anew. And it did so in a new way : by *observing nature* before attempting to explain it.[12]

A taste for objective knowledge[13] reappeared, which had existed during the time of the pre-Socratics, but had then vanished for two thousand years, giving way to considerations or peremptory affirmations about man, the reason for his existence, his preeminence in the universe, the way God wanted us to live on earth before the real life in the Hereafter, etc.

After a few intrepid travelers like William of Rubruck (c. 1220, c. 1290), Marco Polo (1254, 1324) or Ibn Battuta (1304, 1368), in the 15th century the Portuguese embarked on distant maritime expeditions along the coasts of West Africa. At the end of the 15th century, people had gone to America, circumnavigated Africa and reached India by sea. And Europe began to dominate the world.[14]

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12. This avoided thinkers of Antiquity, such as Aristotle (-384, -322) who explained that the heaviest objects fell the fastest, that an object had to be constantly pushed to move forward, that the stars moved in circles in the sky, that space could be filled with regular tetrahedra just as the plane could be tiled with equilateral triangles, etc.

13. In epistemology, the concept of “objective knowledge” is *very slippery*. But we are not going to address the issue here.

14. Before the great European navigators, the Chinese had already set up expeditions in the Indian Ocean, led by Admiral Zheng He (1371, 1433), with much larger ships and much larger squadrons than those of Christopher Columbus or Vasco da Gama. But the fourth emperor of the Ming dynasty had abruptly put an end to them.

Navigation on ocean going vessels required a better understanding of the movement of the stars, because, when one was far from the coast, they were the only points available to find one's bearings at sea (and also in the Sahara desert). Thus the mathematicians and scientists of the 16th century were called upon. They developed new tools in geometry in space and on the sphere. They also understood the importance of *functions of time*, that is, where the independent variable is time, and the dependent variable or variables describe a position somewhere.

In the early 17th century, Kepler formulated his three laws describing the motion of the planets around the Sun. Galileo (1564, 1642) studied the motion of a body in free fall, or sliding or rolling on an inclined plane (so that it does not go too fast). He discovered among other things that all bodies on which air resistance is negligible fall at the same speed.[15] Finally, Newton developed the theory of dynamics and the theory of universal gravitation, which allowed him to reach via pure mathematical calculation the laws that Kepler had found by the careful examination of astronomical data collected by Tycho Brahe (1546, 1601)

In short, the study of functions was launched. And they very quickly became a fundamental tool of mathematics.

Continuous functions

The vast majority of functions that mathematicians and physicists work with are continuous, that is to say, one can draw their graph without lifting the pencil from the sheet of paper.

For a long time, this definition was sufficient. But at the beginning of the 19th century, this was no longer the case. It was necessary to give a more precise definition of what is meant by a continuous function. People like the German-Italian-Czech mathematician Bernard Bolzano (1781, 1848) had begun to understand that the line of numbers was more complicated than it seemed. It had been known since the Greeks that there were irrational numbers, but a more precise definition of irrational numbers was needed than "these are the numbers that are not fractions".

Generally speaking, a definition that says what something is not is a bit insufficient to understand what the thing is :-)

15. This is easy to explain, because a heavy body that breaks into two in the air does not suddenly slow down. But it is even better to verify it experimentally. See the joint fall of a feather and a hammer on the Moon `https://www.youtube.com/watch?v=KDp1tiUsZw8`

It is Cauchy who provided a workable definition of what a continuous function is. And he did so, in a way that is a bit surprising : not by describing the overall behavior of the function on a segment, but by specifying how it should behave at each point of the segment on which it is continuous. In other words, continuity is defined first at a point.

Definition : *A function f from $\mathbb{R}$ to $\mathbb{R}$ is continuous at a given point c*[16]*, if for any sequence of numbers x_n that converges to c, the sequence of numbers $f(x_n)$ converges to $f(c)$.*

Then, by definition too, a function f is continuous on a segment $[a,\ b]$ if it is continuous at all points of $[a,\ b]$.

It is intuitively clear that we can draw the graph of such a function on the interval $[a,\ b]$ without lifting the pencil. However, if the function makes a sudden jump in the ordinate from one value to another at a certain abscissa d, the pencil will need to be lifted.

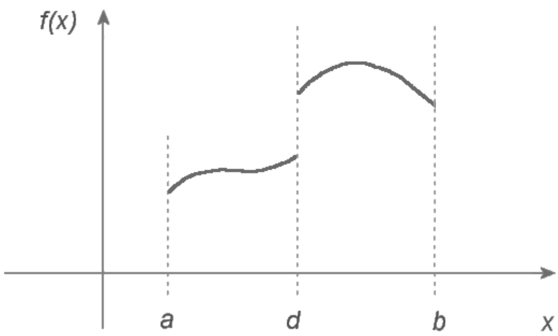


Figure 4.2 : Continuous function on the segment $[a,\ b]$, except at the point d. Note that we have not given on this graph the value of the function at the abscissa d. And whether the segment $a\ b$ is closed or open is not important for the moment (it will be later).

These observations, however, remain in the realm of intuition. We will make them more rigorous – which will lead us once again, and slightly bending our promise of the previous lesson, to briefly discuss topology. We will also present at the end of the lesson some examples of teratological functions that mathematicians keep on their shelves like specimens in jars of formalin.

16. That is, at a given real number c which the independent variable can take, on the x-axis, in the domain of definition of the function.

Examples

Example 1 : The equation $f(x) = ax + b$ defines a continuous function.

Indeed, if x_n is a sequence of numbers converging to l, the corresponding sequence of $f(x_n)$ tends to $al + b$.

Let us give a thorough proof with the help of a little "epsilonite". We want to show that

$$\forall \epsilon,\ \exists N \text{ such that } n > N \implies |f(x_n) - f(l)| < \epsilon$$

We shall use the fact that x_n converges to l. This translates to

$$\forall \eta,\ \exists M \text{ such that } n > M \implies |x_n - l| < \eta$$

Choose $\eta = \dfrac{\epsilon}{|a|}$. Then by going far enough with the index n, we are sure that we will have, for all the n greater than a given M (dependent on η),

$$|x_n - l| < \frac{\epsilon}{|a|} \tag{4.6}$$

This inequality is equivalent to

$$|a|\ |x_n - l| < \epsilon$$

or

$$|ax_n - al| < \epsilon \tag{4.7}$$

In other words, we have just demonstrated that whatever positive number ϵ we pick, however small it may be, by going far enough in the indexes, we will have $f(x_n)$ at a distance of $f(l)$ smaller than ϵ. With the symbolism of epsilonite, this is written

$$\forall \epsilon,\ \exists M \text{ such that } n > M \implies |f(x_n) - f(l)| < \epsilon$$

Q.E.D.

Note that we have just shown that a straight line is a continuous curve. But this last assertion is informal, while our demonstration with ϵ and η is rigorous.

Example 2 : The equation $f(x) = ax^2+bx+c$ defines a continuous function.

We want to show that if x_n is a sequence of numbers tending to l, then the corresponding sequence of $f(x_n)$ tends to $al^2 + bl + c$.

We will again start from the fact that

$$\lim_{n\to+\infty} x_n = l \tag{4.8}$$

That is to say : $\forall\eta$, $\exists M$ such that $n > M \Longrightarrow |x_n - l| < \eta$

There are several possible paths to arrive at

$$\forall\epsilon,\ \exists N \text{ such that } n > N \Longrightarrow |(ax_n^2 + bx_n + c) - (al^2 + bl + c)| < \eta$$

or, more simply,

$$\forall\epsilon,\ \exists N \text{ such that } n > N \Longrightarrow |ax_n^2 + bx_n - al^2 + bl| < \eta$$

To begin, note that

$$|ax_n^2 + bx_n - al^2 + bl| \;\leq\; |ax_n^2 - al^2| + |bx_n - bl| \tag{4.9}$$

We already know that by going far enough in n, the second term on the right-hand side can be made as small as we want. It is therefore sufficient to show that this is also true for $|ax_n^2 - al^2|$.

Let us write x_n in the form $x_n = l + \eta_n$. Then

$$x_n^2 = l^2 + 2l\eta_n + \eta_n^2 \tag{4.10}$$

Since l is a given real number (it is not an "infinite number"), by going far enough with the index n we can make the difference between x_n^2 and l^2 as small as we want in absolute value. And this is also true for ax_n^2.

Once ϵ is chosen, let's take an index n large enough so that on the one hand $|bx_n - bl| < \epsilon/2$ and on the other hand $|ax_n^2 - al^2| < \epsilon/2$. We know that this is possible for $n > M$ large enough. Then the right-hand side of (4.9) will be smaller than ϵ.

That completes the demonstration of

$$\lim_{n\to+\infty} x_n = l \quad\Longrightarrow\quad \lim_{n\to+\infty} ax_n^2 + bx + c = al^2 + bl + c$$

Q.E.D

This time, we have just shown that a parabola is a continuous curve.

Exercise 4.5 : Without going into all the epsilonitis, explain the key points of the demonstration that a polynomial function of degree n is a continuous function.

We can similarly prove that the trigonometric functions, $\sin x$ and $\cos x$ are continuous on all $\mathbb{R}$. We recall that the second is simply the first shifted by $\pi/2$ to the left, that is

$$\cos x = \sin(x + \pi/2)$$

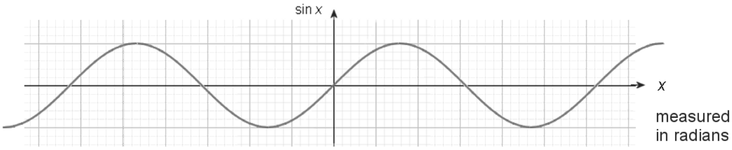


Figure 4.3 : Sine function.

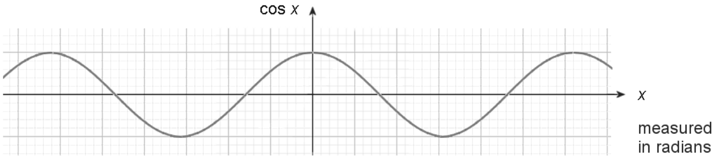


Figure 4.4 : Cosine function.

The rigorous demonstration of the continuity of the function $\sin x$, with epsilonite, is based on the geometric definition of the sine of an angle. Without being complicated, it would not provide any interesting insight, and we are not going to do it. Let us simply observe that both graphs can be drawn without lifting the pen.

The case of the function $\tan x$ presents a particularity. The tangent function is continuous on all $\mathbb{R}$ except at the points

$$\ldots \quad -\frac{3\pi}{2}, \quad -\frac{\pi}{2}, \quad \frac{\pi}{2}, \quad \frac{3\pi}{2}, \quad \frac{5\pi}{2}, \quad \ldots$$

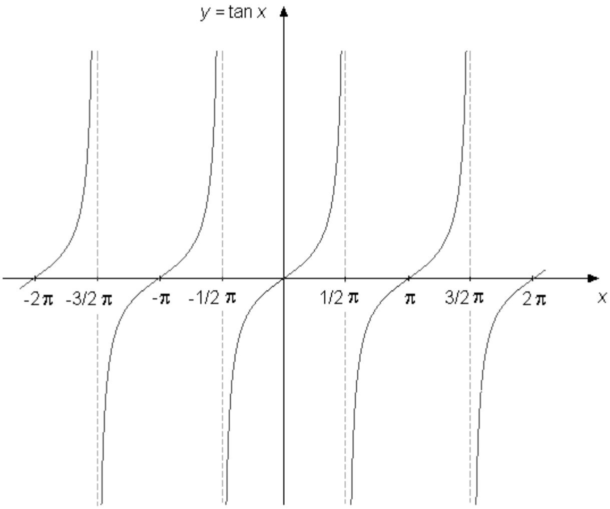


Figure 4.5 : Tangent function.

When the independent variable x (i.e. the angle x) approaches $\pi/2$ its tangent becomes infinite, because

$$\tan x = \frac{\sin x}{\cos x} \tag{4.11}$$

And, as we know, this formula is only valid if $\cos x \neq 0$. If you need to refresh your knowledge of trigonometric functions, you can refer to our book *High school mathematics : 10th grade*, Eagle's Beak Press, fall 2024.

In mathematics a little more advanced than 11th grade, we no longer define trigonometric functions using geometric constructions, but using power series. Thus

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \tag{4.12}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... \tag{4.13}$$

The tangent function also has a power series, where the coefficients have a slightly more complicated expression than formulas (4.12) and (4.13).

We can then demonstrate the continuity of the sine, cosine, and tangent of x using the tool of power series. We can even calculate the derivative functions easily – *after* examining the conditions under which a power series can be differentiated term by term. These topics will be covered later, in the next lesson and next year.

In conclusion, most functions that mathematicians and physicists work with are continuous everywhere, except at a few points, as seen with the tangent function.

Intermediate Value Theorem

Theorem : *Given a function $f(x)$ that is continuous on a segment $[a,\ b]$ of the real numbers, then if $f(a)$ is negative and $f(b)$ is positive, there exists at least one point c located between a and b such that $f(c) = 0$.*

Intuitively this is obvious, see figure 4.6.

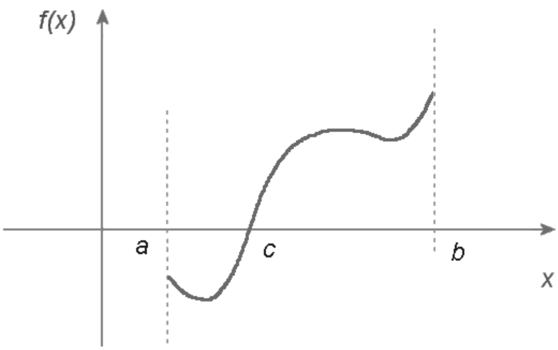


Figure 4.6 : A continuous function on $[a,\ b]$, negative at a and positive at b necessarily intersects the x axis at at least one point c between a and b.

But this is not a proof :-)

Proof of the intermediate value theorem :

Let f be a continuous on $[a,\ b]$. We start by noting that if $f(d)$ is strictly positive, then it is also true in a neighborhood (perhaps very narrow, but nonzero) around d. This is also true to the left of b and to the right of a.

Now consider the set of points x on the segment $[a,\ b]$ where $f(x) < 0$. Let's call it E. This set E is bounded above by b.

Then, by virtue of the fact that the set of real numbers, $\mathbb{R}$, is complete, the set E has a smallest upper bound. Let us call it c.

It is left to the reader to show that $f(c)$ cannot have any value other than zero.

Q.E.D.

The fact that we work in the set $\mathbb{R}$ of real numbers played a fundamental role. If we were working only in the set $\mathbb{Q}$ of rationals, our demonstration would collapse. And indeed, the intermediate value theorem is not true in $\mathbb{Q}$. This is again a consequence of the fact that the rationals may be dense, yet they have holes. And the curve in figure 4.6 could slip through a hole in $\mathbb{Q}$.

For example, on the segment $[0,\ 2]$ *in the set* $\mathbb{Q}$ *of rationals*, the function

$$f(x) = x^2 - 2 \tag{4.14}$$

does not satisfy the intermediate value theorem. Indeed, $f(0) = -2$, $f(2) = +2$, and the function is continuous *in* $\mathbb{Q}$, but there is no rational number c such that $f(c) = 0$.

Hopefully this sheds more light on the topological difference between $\mathbb{Q}$ and $\mathbb{R}$, and why we need to be in the set $\mathbb{R}$ to work conveniently with continuous functions : we obviously want the intermediate value theorem to be true.

Functions with strange behaviors

I promised, to finish, a little tour in the gallery of teratological functions from $\mathbb{R}$ to $\mathbb{R}$.

Here is a first one that is a little bizarre but not too much. It is

$$f(x) = x \sin\left(\frac{1}{x}\right) \tag{4.15}$$

It is defined for all real numbers except $x = 0$ and it is continuous over the entire domain where it is defined.

Its graph is shown in figure 4.7, taken from volume 1, chapter II, of the three-volume book by Aleksandrov, Kolmogorov, and Lavrent'ev mentioned in the suggested reading at the end of lesson 1.

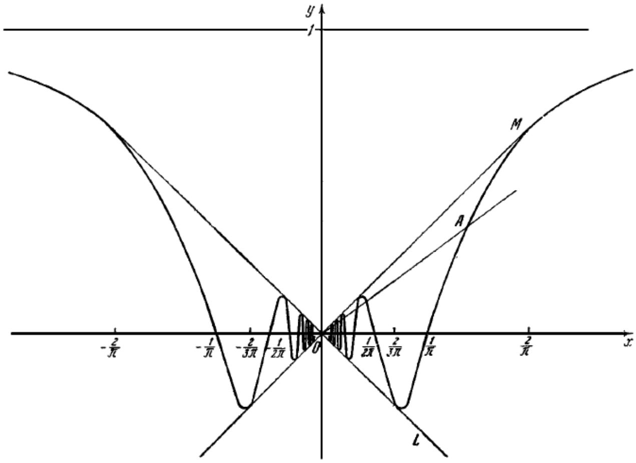


Figure 4.7 : Function $y = x \sin\left(\frac{1}{x}\right)$. It is defined and continuous everywhere on the set of nonzero real numbers. It has the particularity that it can be extended so that it is defined and continuous everywhere : add $f(0) = 0$.

It can be extended by continuity, so that it is defined and continuous everywhere. We only need to assign the value 0 to $f(0)$.

On the other hand, it is teratological in the sense that the slope of its secant to the right and the slope of its secant to the left of zero (we mean by this $[f(x) - f(0)]/[x - 0]$) do not converge towards numbers (even different from each other), but do not cease to oscillate between two values. The secants play a role in the calculation of the derivative function that we will study in the next lesson.

Chapter II of the book by Aleksandrov, Kolmogorov and Lavrent'ev gives other (non-teratological) examples of continuous functions, sometimes with points of discontinuity, as we did above in the examples.

Here is a function that is clearly more teratological than the one above. It is the function of Thomae. It is defined as follows :

- It takes the value 0 on all irrationals.
- It takes the value 1 for $x = 0$.
- And for any other rational x, let p/q be the irreducible fraction representing x with $q > 0$, then the value of the Thomae function at x is $1/q$.

This function is continuous at all irrational points, and discontinuous at all rational points. It cannot be continuous at rational points (or numbers), because there are always irrationals as close as one wants to any rational.

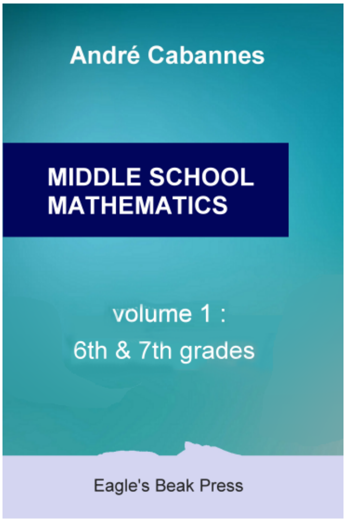
Suggested reading

DESCARTES René, *Discours de la méthode*, Leiden, Netherlands, 1637.

Numerous editions in French and English can be found on the Net. This book is important primarily for two of its three annexes : analytical geometry and optics.

KATZ Victor J., editor, *The Mathematics of Egypt, Mesopotamia, China, India, and Islam : A Sourcebook*, Princeton University Press, 2007.

English titles by André Cabannes



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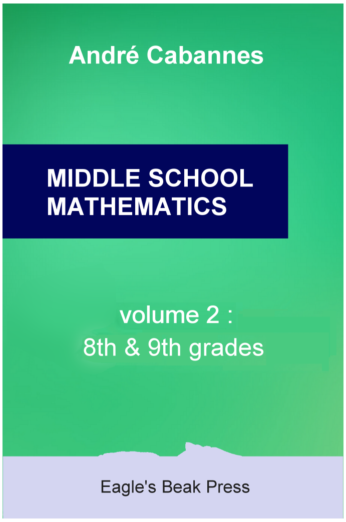

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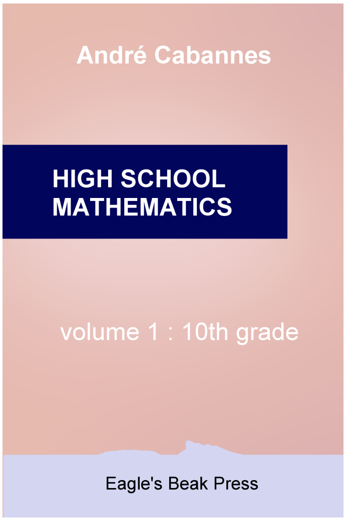

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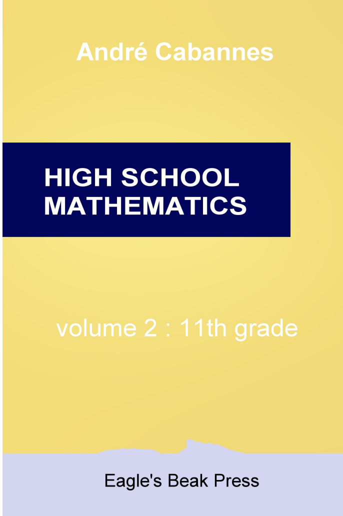

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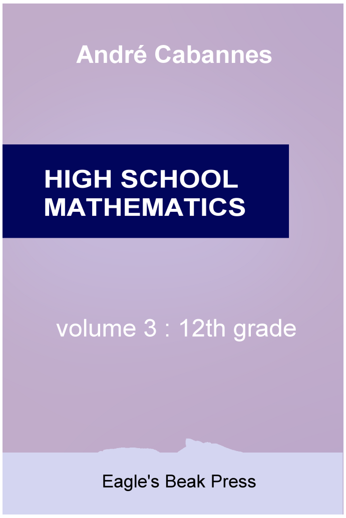

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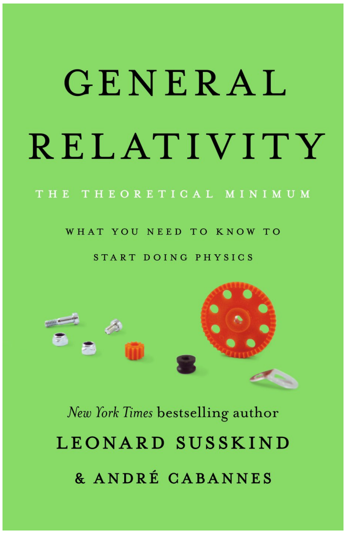

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